

One-parameter groups of operators and discrete Hilbert transforms

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ABSTRACT. We show that the discrete Hilbert transform and the discrete Kak-Hilbert transform are infinitesimal generator of one-parameter groups of operators in ℓ^2 .

1. Introduction

We are concerned with the family of operators $\{T_t\}_{t \in \mathbb{R}}$, initially defined in the space s_0 of complex-valued sequences with compact support as follows:

$$(1.1) \quad (T_t(\vec{a}))_m = \begin{cases} \frac{\sin(\pi t)}{\pi} \sum_{n \in \mathbb{Z}} \frac{a_n}{m - n + t} & \text{if } t \notin \mathbb{Z} \\ (-1)^t a_{m+t} & \text{if } t \in \mathbb{Z}. \end{cases}$$

When t is an integer, $T_t(\vec{a}) = (-1)^t \tau_t(\vec{a})$, where $\tau_k(\vec{a})_m = a_{k+m}$ is the translation; when $t \in (-1, 1)$ these operators can be viewed as discrete versions of the Hilbert transform in $L^2(\mathbb{R})$.

The Hilbert transform

$$\mathcal{H}f(x) = p.v. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x - t} dt,$$

initially defined when $f \in C_0^\infty(\mathbb{R})$, is the archetypal singular integral operator. Discrete analogs of the Hilbert transform have important applications in science and technology. The following operator was introduced by D. Hilbert in 1909.

$$(1.2) \quad (H(\vec{a}))_m = \frac{1}{\pi} \sum_{\substack{n \in \mathbb{Z} \\ n \neq m}} \frac{a_n}{m - n}.$$

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This transformation is not well suited for the applications for reasons that we will discuss in Section 2; the operators

$$T_{\frac{1}{2}}(\vec{a}) = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{a_n}{m - n + \frac{1}{2}}$$

(E.C. Titchmarsh, 1926) and the *Kak-Hilbert transform* (S. Kak, 1970, [8])

$$(1.3) \quad K(\vec{a})(k) = \begin{cases} \frac{2}{\pi} \sum_{n \text{ even}} \frac{a_n}{k - n} & k \text{ odd} \\ \frac{2}{\pi} \sum_{n \text{ odd}} \frac{a_n}{k - n} & k \text{ even} \end{cases}$$

share some of the features of the continuous Hilbert transform and are very relevant in sciences and engineering. We will discuss these operators in Sections 2 and 3. Weighted discrete Hilbert transforms and their connections with problems in complex analysis are discussed in [2].

When $1 \leq p < \infty$, we denote with ℓ^p the space of complex-valued p-summable sequences, i.e.,

$$\ell^p = \left\{ \vec{a} = (a_j)_{j \in \mathbb{Z}} : \|\vec{a}\|_{\ell^p} = \left(\sum_{j \in \mathbb{Z}} |a_j|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

ℓ^∞ is the space of bounded sequences equipped with the norm $\|\vec{a}\|_{\ell^\infty} = \sup_{m \in \mathbb{Z}} |a_m|$.

Our main result is the following

THEOREM 1.1. *The family $\{T_t\}_{t \geq 0}$ defined in (1.1) is a strongly continuous group of isometries in ℓ^2 ; its infinitesimal generator is πH , where H is the operator defined in (1.2).*

To prove Theorem 1.1 we will prove that $T_s \circ T_t = T_{s+t}$; (Theorem 4.1); that T_t is an isometry for every $t \in \mathbb{R}$ (Theorem 4.5); that for every $\vec{a} \in \ell^2$, the application $t \rightarrow T_t(\vec{a})$ is continuous in \mathbb{R} (Theorem 4.3); and finally that, for every $\vec{a} \in \ell^2$, $\lim_{t \rightarrow 0} \frac{T_t(\vec{a}) - \vec{a}}{t} = \pi H(\vec{a})$ (Theorem 4.6).

The proofs of these results are elementary and use only the identities in Section 2.3. Theorem 4.5 seem to be known, but we could not find references in the literature. Some of the results in Section 4 can also be proved in the framework of the theory of Toeplitz operators ¹.

In Section 3 we describe the properties of the Kak-Hilbert transform and we prove the following

THEOREM 1.2. *Let K be the discrete Kak-Hilbert transform (1.3). Then*

$$U_t = \cos tI + \sin tK = \text{Im}(e^{-it}(I + iK)), \quad t \in \mathbb{R}.$$

is a strongly continuous group of operators in ℓ^2 generated by K .

¹We are indebted to I. Verbitsky for this remark

In Sections 2.1 and 3 we discuss the $\ell^p - \ell^p$ boundness of the operators H , T_t and K for $1 < p < \infty$. It is noted in [11] that the operators T_t and H (and in general, every operator $L : \ell^p \rightarrow \ell^p$ in the form of $L(\vec{a})_m = \sum_n a_{m-n} c_n$, with $(c_n)_{n \in \mathbb{Z}} \in \ell^\infty$) can be associated to a Fourier multiplier operator acting on functions on the real line. Indeed, we can associate to L the operator $\tilde{L} : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$

$$\tilde{L}f(x) = \sum_{n \in \mathbb{Z}} f(x-n)c_n = \int_{\mathbb{R}} \hat{f}(y)m(y)e^{2\pi ixy} dy$$

where $m(y)$ is the periodic function on \mathbb{R} whose Fourier coefficients are the c_n 's, and $\hat{f}(y) = \int_{\mathbb{R}} f(x)e^{2\pi ixy} dx$ is the Fourier transform of $f(x)$. For example, it is not too difficult to verify that the multiplier associated to the Kak-Hilbert transform (1.3) is the "square wave" function that coincides with $m(x) = i \operatorname{sgn}(x)$ in $(-\frac{1}{2}, \frac{1}{2})$.

It is proved in [11] that the $\ell^p \rightarrow \ell^p$ operator norm of L is the same as the $L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ operator norm of \tilde{L} . In short,

$$(1.4) \quad |||L|||_{\ell^p} = |||\tilde{L}|||_{L^p}.$$

Since the $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ norm of a multiplier operator is the $L^\infty(\mathbb{R})$ norm of the multiplier (see e.g. [13]) from (1.4) follows that $|||L|||_{\ell^2} = \sup_{x \in \mathbb{R}} |m(x)|$. The evaluation of the $L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ norm of multiplier operators is often a very difficult problem, but the equivalence (1.4) can be used to produce an upper bound for the $\ell^p \rightarrow \ell^p$ norm of L . Indeed, we will prove in Section 3 (Theorems 3.2 and 3.1) that $|||\mathcal{H}|||_{L^p} \leq |||H|||_{\ell^p} \leq |||K|||_{\ell^p} = |||T_{\frac{1}{2}}|||_{\ell^p}$. We conjecture that these norms are equal for all values of $p \in (1, \infty)$.

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2. Preliminaries

2.1. The Hilbert transform. The Hilbert transform

$$\mathcal{H}f(x) = p.v. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt,$$

initially defined when $f \in C_0^\infty(\mathbb{R})$, is an important singular integral operator. We refer the reader to the excellent [5] for an introduction to the Hilbert transform and singular integrals.

The Hilbert transform satisfies the identity $\mathcal{H} \circ \mathcal{H}(f) = -f$, which implies that \mathcal{H} is an isometry in $L^2(\mathbb{R})$.

When f is real-valued, $f + i\mathcal{H}f$ extends to an holomorphic function in the upper complex half-plane. This fundamental property of the Hilbert transform has been used by S. Pichoridis [9] to evaluate the best constant in the inequality of M. Riesz:

$$(2.1) \quad |||\mathcal{H}f|||_{L^p(\mathbb{R})} \leq n_p |||f|||_{L^p(\mathbb{R})}, \quad f \in C_0^\infty(\mathbb{R}).$$

Here, $1 < p < \infty$, and $n_p = \max\{\tan(\pi/2p), \cot(\pi/2p)\}$. See also [4] for a short proof of Pichoridis' result.

Discrete versions of the Hilbert transform have a variety of applications in signal representation and processing. See e.g. [10] and the references cited there.

To the best of our knowledge, the $\ell^2 \rightarrow \ell^2$ norm of the operator H defined in (1.2) has been estimated for the first time by D. Hilbert who in 1909 proved the inequality

$$(2.2) \quad \sum_{m \in \mathbb{Z}} \sum_{\substack{n \in \mathbb{Z} \\ n \neq m}} \frac{a_n b_m}{m - n} \leq c \left(\sum_{n \in \mathbb{Z}} |a_n|^2 \right)^{\frac{1}{2}} \left(\sum_{m \in \mathbb{Z}} |b_m|^2 \right)^{\frac{1}{2}}$$

with a constant $c > \pi$.² Three years later, Shur [12] proved that $c = \pi$ is the best possible constant in the inequality (2.2), or equivalently that 1 is the $\ell^2 \rightarrow \ell^2$ operator norm of H . See [3] for an elegant elementary proof of Shur's inequality.

H is not an isometry in ℓ^2 . Indeed, the proof in [3] shows that the equality $\|H(\vec{a})\|_{\ell^2} = \|\vec{a}\|_{\ell^2}$ only holds when $\vec{a} = 0$. Also, it is not true in general that $H \circ H(\vec{a}) = -\vec{a}$.

The operators $T_{\frac{1}{2}}$ is a good analog of the continuous Hilbert transform. By Theorem 1.1, $T_{\frac{1}{2}}$ is an isometry in ℓ^2 and satisfies $T_{\frac{1}{2}} \circ T_{\frac{1}{2}}(\vec{a}) = -\tau_1(\vec{a})$.

The Kak-Hilbert transform defined in (1.3) can be viewed as a "reduced" discrete Hilbert transform (1.2): if let $\chi_e : \ell^2 \rightarrow \ell^2$ be such that $\chi_e(\vec{a})_n = a_n$ when n is even and $\chi_e(\vec{a})_n = 0$ when n is odd, and we let $\chi_o(\vec{a}) = \vec{a} - \chi_e(\vec{a})$, we can easily verify that

$$(2.3) \quad K(\vec{a}) = 2(\chi_o \circ H \circ \chi_e(\vec{a}) + \chi_e \circ H \circ \chi_o(\vec{a})).$$

S. Kak proved in [8] that $K \circ K(\vec{a}) = -\vec{a}$, from which follows that K is an isometry in ℓ^2 .

It is proved in [6] that H is bounded in ℓ^p for $1 < p < \infty$, and in [11], Theorem 4.3, that the $\ell^p \rightarrow \ell^p$ operator norm of H is $\geq n_p$, where n_p is the constant in (2.1). Equality is proved for special values of p .

It is also proved in [11] that the operators T_t are bounded in ℓ^p for $1 < p < \infty$, and $\|T_t\|_{\ell^p} \geq \|\cos(\pi t)\mathcal{I} + \sin(\pi t)\mathcal{H}\|_{L^p}$, where $\mathcal{I}f = f$. The evaluation of the $\ell^p \rightarrow \ell^p$ operator norms of H and T_t is a tantalizing long-standing open problem.

2.2. Groups and semigroups of operators. Let X be a Banach space with norm $\|\cdot\|$ and let $\mathcal{L}(X)$ be the collection of linear and bounded operators on X . A *one parameter group of operators* is a mapping $U : \mathbb{R} \rightarrow \mathcal{L}(X)$ such that (a) $U(0)$ is the identity operator in $\mathcal{L}(X)$ and (b) $U(s) \circ U(t) = U(s+t)$ whenever $s, t \in \mathbb{R}$. In particular $U(-s) = U^{-1}(s)$.

²The original proof first appeared in Weyl's [15] doctoral dissertation in 1908.

A *semigroup* is a mapping $U : [0, \infty) \rightarrow \mathcal{L}(X)$ that satisfies (a) and (b) whenever $s, t \geq 0$.

We say that a group (or semigroup) U is *strongly continuous* if $\lim_{t \rightarrow t_0} \|U(t)(x) - U(t_0)(x)\| = 0$ for every $x \in X$. When U is a semigroup, we also require that $\lim_{t \rightarrow 0^+} \|U(t)(x) - x\| = 0$. We say that U is *contractive* if $\|U(t)(x)\| \leq \|x\|$ for every $x \in X$ and every $t \in \mathbb{R}$ (or: for every $t \geq 0$ if U is a semigroup).

The *infinitesimal generator* A of a strongly continuous group (or semigroup) $U(t)$ can be introduced as the operator defined by

$$(2.4) \quad A(x) =: \frac{d}{dt}U|_{t=0} = \lim_{h \rightarrow 0^+} \frac{U(h)(x) - x}{h}, \quad x \in D(A)$$

where $D(A)$ is the set of all $x \in X$ for which the above limit exists. Using the strong continuity of $U(t)$, it is possible to prove that $D(A)$ is dense in X . It can also be proved that the equation below is valid for every $x \in D(A)$:

$$(2.5) \quad U(t)(x) =: e^{tA(x)} = \sum_{n=0}^{\infty} \frac{A^{(n)}(x)t^n}{n!}$$

where $A^{(n)}$ denotes the iterated compositions of A .

The Hille-Yosida theorem gives necessary and sufficient conditions for an operator A whose domain is dense in X to be the infinitesimal generators of a contractive semigroup.

THEOREM 2.1. *Let A be a linear operator defined on a linear subspace $D(A)$ of a Banach space X . Then, A is the infinitesimal generator of a contractive semigroup if and only if*

- i) $A - \lambda I$ is invertible for every $\lambda \in (0, \infty)$,
- ii) $\|(A - \lambda I)^{-1}(x)\| < \frac{\|x\|}{\lambda}$ for every $x \in D(A)$ and $\lambda > 0$.

The Hille-Yosida theorem is fundamental in the applications to partial differential equations: indeed, if A is the infinitesimal generator of a semigroup (or group) $U(t)$ in a Banach space X , the vector function $u(t) = U(t)(u_0)$ solves the abstract initial value problem

$$\begin{cases} u'(t) = Au(t) & t > 0 \ (t \in \mathbb{R}) \\ u(0) = u_0 \end{cases}$$

for any given initial value $u_0 \in D(A)$.

We refer the reader to the classical textbooks [7] and [14] for more applications and results.

2.3. Partial fraction decomposition. The partial fraction expression of the cotangent function was proven by Euler in his *Introductio in Analysis Infinitorum* (1748) for every non-integer x , and is regarded as one

of the most interesting formula involving elementary functions:

$$(2.6) \quad \pi \cot(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{x+n} + \frac{1}{x-n}.$$

An elegant proof of this identity can be found in [1], pg. 149. Using (2.6), we can easily prove the following identity, which is valid for every non-integers $u, v \in \mathbb{R}$, with $u \neq v$,

$$(2.7) \quad \sum_{m=-\infty}^{\infty} \frac{1}{(m-u)(m-v)} = \frac{\pi(\cot(\pi v) - \cot(\pi u))}{u-v}.$$

We will also use the following well known identities: when d is not an integer,

$$(2.8) \quad \sum_{n \in \mathbb{Z}} \frac{1}{(n+d)^2} = \pi^2 \csc^2(d\pi)$$

and when d is an integer,

$$(2.9) \quad \sum_{\substack{n \in \mathbb{Z} \\ n \neq -d}} \frac{1}{(n+d)^2} = \frac{\pi^2}{3}.$$

3. The Kak-Hilbert transform

As recalled in Section 2.1, the Kak-Hilbert transform (1.3) shares a remarkable number of properties with the continuous Hilbert transform. Recalling the definition of χ_e and χ_o from section 2.1, we can easily verify that

$$(3.1) \quad \begin{aligned} K(\chi_o(\vec{a}))_{2m} &= \frac{2}{\pi} \sum_{n \in \mathbb{Z}} \frac{a_{2n+1}}{2m-2n-1} = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{a_{2n+1}}{(m-1)-n+\frac{1}{2}} \\ &= \tau_{-1} T_{\frac{1}{2}}(\tau_1 \delta_2(\vec{a}))_m \end{aligned}$$

where we have left $\delta_2((a_j)_{j \in \mathbb{Z}}) = (a_{2j})_{j \in \mathbb{Z}}$ and $\tau_k((b_j)_{j \in \mathbb{Z}}) = (b_{j+k})_{j \in \mathbb{Z}}$. Similarly, we prove that

$$(3.2) \quad K(\chi_e(\vec{a}))_{2m+1} = T_{\frac{1}{2}}(\delta_2(\vec{a}))_m.$$

Note also that $K(\chi_e(\vec{a}))_h = 0$ when h is even and $K(\chi_o(\vec{a}))_h = 0$ when h is odd. Therefore, for every $p > 0$

$$(3.3) \quad \sum_{m \in \mathbb{Z}} |K(\vec{a})_m|^p = \sum_{m \in \mathbb{Z}} |K(\chi_e(\vec{a}))_{2m}|^p + \sum_{m \in \mathbb{Z}} |K(\chi_o(\vec{a}))_{2m+1}|^p.$$

We prove the following

THEOREM 3.1. *For every $1 < p < \infty$*

$$(3.4) \quad |||K|||_{\ell^p} = |||T_{\frac{1}{2}}|||_{\ell^p}.$$

PROOF. Let t_p be the $\ell^p \rightarrow \ell^p$ operator norm of $T_{\frac{1}{2}}$. By (3.1), (3.2) and (3.3)

$$\begin{aligned} \|K(\vec{a})\|_{\ell^p}^p &= \sum_{m \in \mathbb{Z}} |K(\chi_e(\vec{a}))_{2m}|^p + \sum_{m \in \mathbb{Z}} |K(\chi_o(\vec{a}))_{2m+1}|^p, \\ &= \sum_{m \in \mathbb{Z}} |T_{\frac{1}{2}}(\delta_2(\vec{a}))_m|^p + \sum_{m \in \mathbb{Z}} |\tau_{-1}T_{\frac{1}{2}}(\tau_1(\delta_2(\vec{a})))_m|^p \\ &\leq t_p^p (\|\delta_2(\vec{a})\|_{\ell^p}^p + \|\tau_1(\delta_2(\vec{a}))\|_{\ell^p}^p) = t_p^p \|\vec{a}\|_{\ell^p}^p \end{aligned}$$

from which follows that $\|K\|_{\ell^p} \leq t_p$.

Let us show that $\|K(\vec{a})\|_{\ell^p} \geq t_p$. We let $\mathcal{E} = \{\vec{a} \in \ell^p : \chi_o(\vec{a}) = \vec{0}\}$ and observe that for every $\vec{a} \in \mathcal{E}$, we have that $\|\vec{a}\|_{\ell^p} = \|\delta_2(\vec{a})\|_{\ell^p}$. In view of $K(\chi_e(\vec{a})) = T_{\frac{1}{2}}(\delta_2(\vec{a}))$, we can write the following chain of inequalities:

$$\|K\|_{\ell^p} \geq \sup_{\vec{a} \in \mathcal{E}} \frac{\|K(\vec{a})\|_{\ell^p}}{\|\vec{a}\|_{\ell^p}} = \sup_{\vec{a} \in \mathcal{E}} \frac{\|T_{\frac{1}{2}}(\delta_2(\vec{a}))\|_{\ell^p}}{\|\delta_2(\vec{a})\|_{\ell^p}} = \sup_{\vec{b} \in \ell^p} \frac{\|T_{\frac{1}{2}}(\vec{b})\|_{\ell^p}}{\|\vec{b}\|_{\ell^p}} = t_p.$$

as required. \square

We let

$$(3.5) \quad \tilde{K}(\vec{a})_m = (2H - K)(\vec{a}) = 2(\chi_e \circ H \circ \chi_e(\vec{a}) + \chi_o \circ H \circ \chi_o(\vec{a})).$$

Thus, $\tilde{K}(\chi_e(\vec{a}))_h = 0$ when h is odd and $\tilde{K}(\chi_o(\vec{a}))_h = 0$ when h is even, and for every $p > 0$

$$(3.6) \quad \sum_{m \in \mathbb{Z}} |\tilde{K}(\vec{a})_m|^p = \sum_{m \in \mathbb{Z}} |\tilde{K}(\chi_e(\vec{a}))_{2m}|^p + \sum_{m \in \mathbb{Z}} |\tilde{K}(\chi_o(\vec{a}))_{2m+1}|^p.$$

We can easily verify that

$$(3.7) \quad \tilde{K}(\chi_e(\vec{a}))_{2m} = H(\delta_2(\vec{a}))_m, \quad \tilde{K}(\chi_o(\vec{a}))_{2m+1} = H(\tau_1 \delta_2(\vec{a}))_m.$$

We prove the following

THEOREM 3.2. *For every $1 < p < \infty$,*

$$(3.8) \quad \|\tilde{K}\|_{\ell^p} = \|H\|_{\ell^p}$$

and

$$(3.9) \quad \|K\|_{\ell^p} \geq \|H\|_{\ell^p}.$$

PROOF. The proof of (3.8) is similar to that of (3.4). To prove (3.9), we observe that $K = 2H - \tilde{K}$, and so

$$\|K\|_{\ell^p} \geq \left| 2\|H\|_{\ell^p} - \|\tilde{K}\|_{\ell^p} \right| = \|H\|_{\ell^p}.$$

\square

Remark. Recall that the $L^p(\mathbb{R}) - L^p(\mathbb{R})$ operator norm of the Hilbert transform is the constant n_p defined in (2.1) and that $|||H|||_{\ell^p} \geq n_p$; by Theorems 3.2 and 3.1,

$$n_p \leq |||\tilde{K}|||_{\ell^p} = |||H|||_{\ell^p} \leq |||K|||_{\ell^p} = |||T_{\frac{1}{2}}|||_{\ell^p}.$$

It is conjectured in [11] that $|||T_{\frac{1}{2}}|||_{\ell^p} = n_p$. If this conjecture is proved, then also the operator norms of H , K and \tilde{K} equal n_p .

PROOF OF THEOREM 1.2. The semigroup generated by K is the operator $U_t = e^{tK} = \sum_{n=0}^{\infty} \frac{t^n}{n!} K^{(n)}$ where $K^{(n)}$ is the n -times composition of K with itself. Recalling that $K \circ K = -I$, we obtain

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} K^{(n)} = I + Kt - \frac{It^2}{2} - \frac{Kt^3}{3!} + \dots = I \cos t + K \sin t.$$

as required. \square

Remark. U_t is not an isometry in ℓ^2 . Indeed,

$$\begin{aligned} ||U_t(\vec{a})||_{\ell^2}^2 &= ||\vec{a} \cos t + K(\vec{a}) \sin t||_{\ell^2}^2 \\ &= |\cos t|^2 ||\vec{a}||_{\ell^2}^2 + |\sin t|^2 ||K(\vec{a})||_{\ell^2}^2 + 2 \sin t \cos t \operatorname{Re} \langle \vec{a}, K(\vec{a}) \rangle \\ &= ||\vec{a}||_{\ell^2}^2 + \sin(2t) \operatorname{Re} \langle \vec{a}, K(\vec{a}) \rangle \end{aligned}$$

and we may have $\operatorname{Re} \langle \vec{a}, K(\vec{a}) \rangle \neq 0$.

4. Proof of Theorem 1.1

The proof of Theorem 1.1 follows from several theorems and lemmas. Some of the results in this section (in particular Theorem 4.5) seem to be known, but we could not find proof of these results in the literature.

First of all, we show that T_t is a semigroup.

THEOREM 4.1. *for every $s, t \in \mathbb{R}$ and for every $\vec{a} \in \ell^2$,*

$$(4.1) \quad T_d \circ T_s(\vec{a}) = T_{s+d}(\vec{a}).$$

In particular, $T_s^{-1}(\vec{a}) = T_{-s}(\vec{a})$.

PROOF. It is enough to prove the theorem for sequences $\vec{a} \in s_0$, the space of complex-valued sequences with compact support, because s_0 is dense in ℓ^2 .

The identity (4.1) is clearly true when s and d are both integers. When s is an integer and d is not integer

$$\begin{aligned} (T_s T_d(\vec{a}))_k &= (-1)^s \frac{\sin(\pi d)}{\pi} \sum_{m \in \mathbb{Z}} \frac{a_m}{(k+s) - m + d} \\ &= \frac{\sin(\pi(d+s))}{\pi} \sum_{m \in \mathbb{Z}} \frac{a_m}{k - m + (d+s)} = T_{d+s}(\vec{a}). \end{aligned}$$

Suppose that s, d and $s+d$ are not integers; let $\vec{a} \in s_0$. We can exchange the order of summation and make use of the identity (2.7) to show that

$$\begin{aligned}
(T_d T_s(\vec{a}))_k &= \frac{\sin(\pi s) \sin(\pi d)}{\pi^2} \sum_{n \in \mathbb{Z}} a_n \sum_{m \in \mathbb{Z}} \frac{1}{(k-m+d)(m-n+s)} \\
&= \frac{\sin(\pi s) \sin(\pi d)}{\pi} \sum_{n \in \mathbb{Z}} a_n \frac{\cot(\pi(d-k)) - \cot(\pi(-n-s))}{n-k+d+s} \\
&= \frac{\sin(\pi s) \sin(\pi d)}{\pi} (\cot(\pi d) + \cot(\pi s)) \sum_{n \in \mathbb{Z}} \frac{a_n}{n-k+d+s} \\
&= \frac{\sin(\pi(s+d))}{\pi} \sum_{n \in \mathbb{Z}} \frac{a_n}{n-k+d+s} = T_{s+d}(\vec{a}).
\end{aligned}$$

as required.

When s, d are not integers and $s+d$ is an integer the proof is similar. \square

We prove that T_t is strongly continuous. We start with the following

LEMMA 4.2. *Let H be as in (1.2). For a given $d \in (-1, 1)$ and for every $\vec{a} \in \ell^2$, we let $H_d(\vec{a})_m = \frac{1}{\pi} \sum_{\substack{n \in \mathbb{Z} \\ n \neq m}} \frac{a_n}{n-m+d}$. We have,*

$$\lim_{d \rightarrow 0} \|H(\vec{a}) - H_d(\vec{a})\|_{\ell^2} = 0.$$

PROOF. Observe that $H_d(\vec{a})_m - H(\vec{a})_m = -\frac{d}{\pi} \sum_{\substack{n \in \mathbb{Z} \\ n \neq m}} \frac{a_n}{(m-n+d)(m-n)}$ is the convolution of \vec{a} and $\vec{v} = (\frac{1}{n(n+d)})_{\substack{n \in \mathbb{Z} \\ n \neq 0}}$. Using the identity (2.6) we can easily prove that $\|\vec{v}\|_{\ell^1} = \sum_{n \neq 0} \frac{1}{n(n+d)} = \frac{1-\pi d \cot(\pi d)}{d^2}$. Furthermore, it is easy to verify that $\lim_{d \rightarrow 0} d \|\vec{v}\|_{\ell^1} = \lim_{d \rightarrow 0} \frac{1-\pi d \cot(\pi d)}{d} = 0$. By Young inequality,

$$\lim_{d \rightarrow 0} \|H_d(\vec{a}) - H(\vec{a})\|_{\ell^2} \leq \frac{1}{\pi} \lim_{d \rightarrow 0} d \|\vec{v}\|_{\ell^1} \|\vec{a}\|_{\ell^2} = 0$$

as required. \square

Remark. The proof of Lemma 4.2 shows that H_d is bounded in ℓ^2 . Indeed,

$$\begin{aligned}
\|H_d(\vec{a})\|_{\ell^2} &\leq \|H_d(\vec{a}) - H(\vec{a})\|_{\ell^2} + \|H(\vec{a})\|_{\ell^2} \\
&\leq \left(\frac{1 - \pi d \cot(\pi d)}{\pi d} + 1 \right) \|\vec{a}\|_{\ell^2}.
\end{aligned}$$

We can now prove that T_t is strongly continuous.

THEOREM 4.3. *For every $t_0 \in \mathbb{R}$ and every $\vec{a} \in \ell^2$,*

$$\lim_{t \rightarrow t_0} \|T_t(\vec{a}) - T_{t_0}(\vec{a})\|_{\ell^2} = 0.$$

PROOF. By Theorem 4.1, $(T_t - T_{t_0})(\vec{a}) = T_{t_0}(T_{t-t_0} - I)(\vec{a})$, where I is the identity in ℓ^2 . So, in order to prove the theorem we need only to prove that $\lim_{d \rightarrow 0} \|T_d(\vec{a}) - \vec{a}\|_{\ell^2} = 0$ for every $\vec{a} \in \ell^2$. Indeed, for every $m \in \mathbb{Z}$,

$$\begin{aligned} T_d(\vec{a})_m - a_m &= \frac{\sin(\pi d)}{\pi} \sum_{n \in \mathbb{Z}} \frac{a_n}{n - m + d} - a_m \\ &= \sin(\pi d) H_d(\vec{a})_m + \frac{\sin(\pi d)}{\pi d} a_m - a_m \end{aligned}$$

where H_d is as in Lemma 4.2. Thus,

$$\lim_{d \rightarrow 0} \|T_d(\vec{a}) - \vec{a}\|_{\ell^2} \leq \lim_{d \rightarrow 0} \left(\frac{\sin(\pi d)}{\pi d} - 1 \right) \|\vec{a}\|_{\ell^2} + \lim_{d \rightarrow 0} \sin(\pi d) \|H_d(\vec{a})\|_{\ell^2} = 0.$$

□

We denote with T_t^* the adjoint operator of T_t ; in order to prove that T_t is an isometry, we need the following useful lemma:

LEMMA 4.4. *For every $t \in \mathbb{R}$ and every $\vec{a} \in \ell^2$,*

$$T_t^*(\vec{a}) = T_{-t}(\vec{a}).$$

PROOF. The lemma is trivial when $t \in \mathbb{Z}$; if $t \notin \mathbb{Z}$, and $\vec{a}, \vec{b} \in \ell^2$,

$$\begin{aligned} \langle T_t(\vec{a}), \vec{b} \rangle &= \frac{\sin(\pi t)}{\pi} \sum_{m \in \mathbb{Z}} b_m \sum_{n \in \mathbb{Z}} \frac{a_n}{n - m + t} \\ &= - \frac{\sin(\pi t)}{\pi} \sum_{n \in \mathbb{Z}} a_n \sum_{m \in \mathbb{Z}} \frac{b_m}{m - n - t} = \langle \vec{a}, T_{-t} \vec{b} \rangle \end{aligned}$$

as required.

□

THEOREM 4.5. *For every $t \in \mathbb{R}$ and every $a \in \ell^2$,*

$$\|T_t(\vec{a})\|_{\ell^2} = \|\vec{a}\|_{\ell^2}.$$

PROOF. By Lemma 4.4 and Theorem 4.1,

$$\|T_t(\vec{a})\|_{\ell^2}^2 = \langle T_t(\vec{a}), T_t(\vec{a}) \rangle = \langle T_{-t} T_t(\vec{a}), \vec{a} \rangle = \|\vec{a}\|_{\ell^2}^2.$$

□

We are left to prove that πH is the infinitesimal generator of T_t .

THEOREM 4.6. *For every $\vec{a} \in \ell^2$,*

$$\lim_{d \rightarrow 0} \left\| \frac{T_d(\vec{a}) - \vec{a}}{d} - \pi H(\vec{a}) \right\|_{\ell^2} = 0.$$

PROOF. We can write

$$\begin{aligned} \frac{(T_d(\vec{a}) - \vec{a})_m}{d} &= \frac{\sin(\pi d)}{\pi d} \sum_{n \in \mathbb{Z}} \frac{a_n}{m - n + d} - \frac{a_m}{d} \\ &= \frac{\sin(\pi d)}{d} H_d(\vec{a})_m + \frac{a_m}{d} \left(\frac{\sin(\pi d)}{\pi d} - 1 \right). \end{aligned}$$

Thus,

$$\begin{aligned} \left| \frac{(T_d(\vec{a}) - \vec{a})_m}{d} - \pi H(\vec{a})_m \right| &\leq \left| \frac{\sin(\pi d)}{d} H_d(\vec{a})_m - \pi H(\vec{a})_m \right| + \left| \frac{a_m}{d} \right| \left| 1 - \frac{\sin(\pi d)}{\pi d} \right| \\ &\leq \left| \frac{\sin(\pi d)}{d} \right| |H_d(\vec{a})_m - H(\vec{a})_m| + \left| \frac{\sin(\pi d)}{d} - \pi \right| |H(\vec{a})_m| + \left| \frac{a_m}{d} \right| \left| 1 - \frac{\sin(\pi d)}{\pi d} \right|, \end{aligned}$$

and for every $\vec{a} \in \ell^2$, we have

$$\begin{aligned} \left\| \frac{T_d(\vec{a}) - \vec{a}}{d} - \pi H(\vec{a}) \right\|_{\ell^2} &\leq \left| \frac{\sin(\pi d)}{d} \right| \|H_d(\vec{a}) - H(\vec{a})\|_{\ell^2} \\ &\quad + \left| \frac{\sin(\pi d)}{d} - \pi \right| \|H(\vec{a})\|_{\ell^2} + \frac{\|\vec{a}\|_{\ell^2}}{d} \left| 1 - \frac{\sin(\pi d)}{\pi d} \right|. \end{aligned}$$

Since $\lim_{d \rightarrow 0} \frac{\sin(\pi d)}{d} - \pi = \lim_{d \rightarrow 0} \frac{1}{d} \left| \frac{\sin(\pi d)}{\pi d} - 1 \right| = 0$, the inequalities above and Lemma 4.2 yield $\lim_{d \rightarrow 0} \left\| \frac{T_d(\vec{a}) - \vec{a}}{d} - \pi H(\vec{a}) \right\|_{\ell^2} = 0$ as required. \square

4.1. Corollaries. The following Corollaries easily follows from Theorems 4.1 and 4.6 and (2.4) and (2.5).

COROLLARY 4.7. *let T_t be as in (1.1) and H as in (1.2). For every $s \in \mathbb{R}$, we have*

$$\begin{aligned} \text{a) } T_s H &= H T_s = \frac{1}{\pi} \frac{d}{dt} T_t \Big|_{t=s}, \\ \text{b) } T_s &= e^{s\pi H} = \sum_{k=0}^{\infty} \frac{(\pi s)^k}{k!} H^{(k)}. \end{aligned}$$

COROLLARY 4.8. *$U(t, \vec{b}) = T_t(\vec{b})$ is the solution to the initial value problem*

$$\begin{cases} \frac{d}{dt} U(t, \vec{a}) = \pi H(U(t, \vec{a})) & U \in C^1(\mathbb{R}, \ell^2) \\ U(0, \vec{a}) = \vec{b}. \end{cases}$$

Corollary 4.8 may have application to signal processing. The following result is a consequence of the Hille-Yosida theorem.

COROLLARY 4.9. *For every $\lambda > 0$, the operator $\pi H - \lambda I$ is invertible in ℓ^2 , and $\|(\pi H - \lambda I)^{-1}(\vec{a})\|_{\ell^2} < \frac{\|\vec{a}\|_{\ell^2}}{\lambda}$.*

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